

Generalized Uncertainty Relations, Fundamental Length and Density Matrix

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Abstract

It was shown that if in Quantum Theory a fundamental length exists and a well-known measurement procedure is used, then the density matrix at the Planck scale cannot be defined in the usual way, because in this case density matrix trace is strongly less than one. Density matrix must be changed by a progenitrix or as we call it throughout this paper, density pro-matrix. This pro-matrix is a deformed density matrix, which at low energy limit turns to usual one. Below the explicit form of the deformation is described. Implications of obtained results are summarized as well as their application to the interpretation of Information Paradox on the Black Holes.

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1 Introduction

In this paper we will show that if in Quantum Theory there is a fundamental length and a known measurement procedure is used, then the density matrix at the Planck scale cannot be defined in the usual way, because in this case density matrix trace is strongly less than one. Since from Generalized Uncertainty Relations (GUR) follow that there is a fundamental length, then one of the implications of existence of GUR is the fact that in this case density matrix cannot be defined in the usual way. It was shown, that commonly accepted definition of density matrix cannot be used at Planck scale and it is necessary to use density pro-matrix, which appears when Quantum Mechanics with Fundamental Length (QMFL) is considered as a deformation usual Quantum Mechanics (QM). This deformation is described explicitly. It was shown also, that inflationary model contains two different (unitary non-equivalent) Quantum Mechanics: the first one describes nature at the Planck scale and it is QMFL. The second one is obtained as a limit transition from Planck scale to low energy one and it is based on the Heisenberg Uncertainty Relations (UR) [1]. It is QM. The interpretation of obtained results as well as their implications are discussed below, in particular for explaining the information paradox in primordial black holes .

2 General Uncertainty Relations, Fundamental Length and Density Matrix

Using different approaches (String Theory [2], Gravitation [3], Quantum Theory of black holes [4] etc [5]). the authors of numerous papers issued over the last 14-15 years have pointed out that Heisenberg's Uncertainty Relations should be modified. Specifically, a high energy correction has to appear

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha L_p^2 \frac{\Delta p}{\hbar}. \quad (1)$$

Here L_p is the Planck's length: $L_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 1,6 \cdot 10^{-35} \text{ m}$ and $\alpha > 0$ is a constant. In [3] it was shown that this constant may be chosen equal to 1. However, here we will use α as an arbitrary constant without giving it

any definite value. The inequality (1) is quadratic in Δp :

$$\alpha L_p^2 (\Delta p)^2 - \hbar \Delta x \Delta p + \hbar^2 \leq 0, \quad (2)$$

from whence the fundamental length is

$$\Delta x_{min} = 2\sqrt{\alpha} L_p. \quad (3)$$

Since in what follows we proceed only from the existence of fundamental length, it should be noted that this fact was established apart from GUR as well. For instance, from an ideal experiment associated with Gravitational Field and Quantum Mechanics a lower bound on minimal length was obtained in [6], [7] and improved in [8] without using GUR to an estimate of the form $\sim L_p$. Let us to consider equation (3) in some detail. Squaring both its sides, we obtain

$$\overline{(\Delta \hat{X}^2)} \geq 4\alpha L_p^2, \quad (4)$$

Or in terms of density matrix

$$Sp[(\rho \hat{X}^2) - Sp^2(\rho \hat{X})] \geq 4\alpha L_p^2 = l_{min}^2 > 0, \quad (5)$$

where \hat{X} is the coordinate operator. Expression (5) gives the measuring rule used in QM. However, in the case considered here, in comparison with QM, the right part of (5) cannot be done arbitrarily near to zero since it is limited by $l_{min}^2 > 0$, where due to GUR $l_{min} \sim L_p$.

Apparently, this may be due to the fact that QMFL with GUR (1) is unitary non-equivalent to QM with UR. Actually, in QM the left-hand side of (5) can be chosen arbitrary closed to zero, whereas in QMFL this is impossible. But if two theories are unitary equivalent then, the form of their spurs should be retained. Besides, a more important aspect is contributing to unitary non-equivalence of these two theories: QMFL contains three fundamental constants (independent parameters) G , c and \hbar , whereas QM contains only one: \hbar . Within an inflationary model (see [10]), QM is the low-energy limit of QMFL (QMFL turns to QM) for the expansion of the Universe. In this case, the second term in the right-hand side of (1) vanishes and GUR turn to UR. A natural way for studying QMFL is to consider this theory as a deformation of QM, turning to QM at the low energy limit (during the expansion of the Universe after the Big Bang). We will consider precisely this option. However differing from authors of papers [4], [5] and others, we do not deform commutators, but density matrix,

leaving at the same time the fundamental quantum-mechanical measuring rule (5) without changes. Here the following question may be formulated: how should be deformed density matrix conserving quantum-mechanical measuring rules in order to obtain self-consistent measuring procedure in QMFL? For answering to the question we will use the R-procedure. For starting let us to consider R-procedure both at the Planck's energy scale and at the low-energy one. At the Planck's scale $a \approx il_{min}$ or $a \sim iL_p$, where i is a small quantity. Further a tends to infinity and we obtain for density matrix

$$Sp[\rho a^2] - Sp[\rho a]Sp[\rho a] \simeq l_{min}^2 \text{ or } Sp[\rho] - Sp^2[\rho] \simeq l_{min}^2/a^2.$$

Therefore:

1. When $a < \infty$, $Sp[\rho] = Sp[\rho(a)]$ and $Sp[\rho] - Sp^2[\rho] > 0$. Then, $Sp[\rho] < 1$ that corresponds to the QMFL case.
2. When $a = \infty$, $Sp[\rho]$ does not depend on a and $Sp[\rho] - Sp^2[\rho] \rightarrow 0$. Then, $Sp[\rho] = 1$ that corresponds to the QM case.

How should be points 1 and 2 interpreted? How does analysis above-given agree to the main result from [21] ¹? It is in full agreement. Indeed, when state-vector reduction (R-procedure) takes place in QM then, always an eigenstate (value) is chosen exactly. In other words, the probability is equal to 1. As it was pointed out in the above-mentioned point 1 the situation changes when we consider QMFL: it is impossible to measure coordinates exactly since it never will be absolutely reliable. We obtain in all cases a probability less than 1 ($Sp[\rho] = p < 1$). In other words, any R-procedure in QMFL leads to an eigenvalue, but only with a probability less than 1. This probability is as near to 1 as far the difference between measuring scale a and l_{min} is growing, or in other words, when the second term in (1) becomes insignificant and we turn to QM. Here there is not a contradiction with [21]. In QMFL there are not exact coordinate eigenstates (values) as well as there are not pure states. In this paper we do not consider operator properties in QMFL as it was done in [21] but density-matrix properties.

The properties of density matrix in QMFL and QM have to be different. The only reasoning in this case may be as follows: QMFL must differ from

¹"... there cannot be any physical state which is a position eigenstate since a eigenstate would of course have zero uncertainty in position"

QM, but in such a way that in the low-energy limit a density matrix in QMFL must coincide with the density matrix in QM. That is to say, QMFL is a deformation of QM and the parameter of deformation depends on the measuring scale. This means that in QMFL $\rho = \rho(x)$, where x is the scale, and for $x \rightarrow \infty$ $\rho(x) \rightarrow \hat{\rho}$, where $\hat{\rho}$ is the density matrix in QM.

Since on the Planck's scale $Sp[\rho] < 1$, then for such scales $\rho = \rho(x)$, where x is the scale, is not a density matrix as it is generally defined in QM. On Planck's scale we name $\rho(x)$ "density pro-matrix". As follows from the above, the density matrix $\hat{\rho}$ appears in the limit

$$\lim_{x \rightarrow \infty} \rho(x) \rightarrow \hat{\rho}, \quad (6)$$

when GUR (1) turn to UR and QMFL turns to QM.

Thus, on Planck's scale the density matrix is inadequate to obtain all information about the mean values of operators. A "deformed" density matrix (or pro-matrix) $\rho(x)$ with $Sp[\rho] < 1$ has to be introduced because a missing part of information $1 - Sp[\rho]$ is encoded in the quantity l_{min}^2/a^2 , whose specific weight decreases as the scale a expressed in units of l_{min} is going up.

3 QMFL as a deformation of QM

Here we are going to describe QMFL as a deformation of QM using the density pro-matrix formalism. In this context density pro-matrix has to be understood as a deformed density matrix in QMFL. As fundamental deformation parameter we will use $\beta = l_{min}^2/x^2$, where x is the scale.

Definition 1.

Any system in QMFL is described by a density pro-matrix $\rho(\beta) = \sum_i \omega_i(\beta) |i\rangle\langle i|$, where

1. $0 < \beta \leq 1/4$;
2. The vectors $|i\rangle$ form a full orthonormal system;
3. $\omega_i(\beta) \geq 0$ and for all i there is a finite limit $\lim_{\beta \rightarrow 0} \omega_i(\beta) = \omega_i$;
4. $Sp[\rho(\beta)] = \sum_i \omega_i(\beta) < 1$, $\sum_i \omega_i = 1$;

5. For any operator B and any β there is a mean operator B , which depends on β :

$$\langle B \rangle_\beta = \sum_i \omega_i(\beta) \langle i|B|i \rangle.$$

At last, in order to match our definition with the result of section 2 the next condition has to be fulfilled:

$$Sp[\rho(\beta)] - Sp^2[\rho(\beta)] \approx \beta, \quad (7)$$

from which we can find the meaning of the quantity $Sp[\rho(\beta)]$, which satisfies the condition of definition:

$$Sp[\rho(\beta)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \beta}. \quad (8)$$

From point 5. it follows, that $\langle 1 \rangle_\beta = Sp[\rho(\beta)]$. Therefore for any scalar quantity f we have $\langle f \rangle_\beta = f Sp[\rho(\beta)]$. In particular, the mean value $\langle [x_\mu, p_\nu] \rangle_\beta$ is equal to

$$\langle [x_\mu, p_\nu] \rangle_\beta = i\hbar \delta_{\mu,\nu} Sp[\rho(\beta)] \quad (9)$$

We will call density matrix the limit $\lim_{\beta \rightarrow 0} \rho(\beta) = \rho$. It is evident, that in the limit $\beta \rightarrow 0$ we turn to QM. Here we would like to verify, that two cases described above correspond to the meanings of β . In the first case when β is near to $1/4$. In the second one when it is near to zero.

From the definitions given above it follows that $\langle (j \rangle \langle j) \rangle_\beta = \omega_j(\beta)$.

From which the condition of completeness on β is

$\langle (\sum_i |i\rangle \langle i|) \rangle_\beta = \langle 1 \rangle_\beta = Sp[\rho(\beta)]$. The norm of any vector $|\psi\rangle$, assigned to β can be defined as

$\langle \psi|\psi \rangle_\beta = \langle \psi | (\sum_i |i\rangle \langle i|)_\beta | \psi \rangle = \langle \psi | (1)_\beta | \psi \rangle = \langle \psi|\psi \rangle Sp[\rho(\beta)]$,

where $\langle \psi|\psi \rangle$ is the norm in QM, or in other words when $\beta \rightarrow 0$. By analogy, for probabilistic interpretation the same situation takes place in the described theory, but only changing ρ by $\rho(\beta)$.

Some remarks:

- I. The considered above limit covers at the same time Quantum and Classical Mechanics. Indeed, since $\beta = l_{min}^2/x^2 = G\hbar/c^3x^2$, so we obtain:

- a. $(\hbar \neq 0, x \rightarrow \infty) \Rightarrow (\beta \rightarrow 0)$ for QM;
 - b. $(\hbar \rightarrow 0, x \rightarrow \infty) \Rightarrow (\beta \rightarrow 0)$ for Classical Mechanics;
- II. The parameter of deformation β should take the meaning $0 < \beta \leq 1$. However, as we can see from (8), and as it was indicated in the section 2, $Sp[\rho(\beta)]$ is well defined only for $0 < \beta \leq 1/4$. That is if $x = il_{min}$ and $i \geq 2$ then, there is not any problem. At the very point with fundamental length $x = l_{min} \sim L_p$ there is a singularity, which is connected with the appearance of the complex value of $Sp[\rho(\beta)]$, or in other words it is connected with the impossibility of obtain a diagonalized density pro-matrix at this point over the field of real numbers. For this reason definition 1 at the initial point do not has any sense.
- III. We have to consider the question about solutions (7). For instance, one of the solutions (7), at least at first order on β is $\rho^*(\beta) = \sum_i \alpha_i \exp(-\beta)|i\rangle\langle i|$, where all $\alpha_i > 0$ do not depend on β and their sum is equal to 1, that is $Sp[\rho^*(\beta)] = \exp(-\beta)$. Indeed, we can easy verify that

$$Sp[\rho^*(\beta)] - Sp^2[\rho^*(\beta)] = \beta + O(\beta^2). \quad (10)$$

- IV. It is clear, that in the proposed description of states, which have a probability equal to 1, or in others words pure states can appear only in the limit $\beta \rightarrow 0$, or when all states $\omega_i(\beta)$ except one of them are equal to zero, or when they tend to zero at this limit.
- V. We suppose, that all definitions concerning density matrix can be transferred to the described above deformation of Quantum Mechanics (QMFL) changing the density matrix ρ by the density pro-matrix $\rho(\beta)$ and turning then to the low energy limit $\beta \rightarrow 0$. In particular, for statistical entropy we have

$$S_\beta = -Sp[\rho(\beta) \ln(\rho(\beta))]. \quad (11)$$

The quantity S_β , evidently never is equal to zero, since $\ln(\rho(\beta)) \neq 0$ and, therefore S_β may be equal to zero only at the limit $\beta \rightarrow 0$.

4 Some Implications

1. If we carry out a measurement in a defined scale, we cannot consider a density pro-matrix with a precision, which exceed some limit of order $\sim 10^{-66+2n}$, where 10^{-n} is the scale in which the measurement is carried out. In most of the known cases this precision is quite enough for considering density pro-matrix the density matrix. However, at the Planck scale, where Quantum Gravity effects cannot be neglected and energy is of the Planck order the difference between $\rho(\beta)$ and ρ have to be considered.
2. At the Planck scale the notion of Wave Function of the Universe, introduced by J.A. Wheeler and B. deWitt [9] does not work and in this case quantum gravitation effects can be described only with the help of density pro-matrix $\rho(\beta)$.
3. Since density pro-matrix $\rho(\beta)$ depends on the scale in which the measurement is carried out, so the evolution of the Universe within inflation model paradigm [10] is not an unitary process, because, otherwise the probability $p_i = \omega_i(\beta)$ would be conserved.

5 On the problem of information paradox in Black Holes

The results obtained above give us the opportunity for considering again the problem of loss information on Black Holes [11, 12, 13], at least for the case of primordial Black Holes. Indeed, because when we consider these Black Holes the Planck's scale is important, then as it was shown above the entropy of matter observed by a Black Hole at this scale is not equal to zero, as it was confirmed by R. Myers [14]. According to his results a pure state cannot form a Black Hole. In this case it is necessary to reformulate the problem itself, since in all published papers on information paradox up to now the equal to zero entropy at the initial state is equal to non zero one at the final state. It is necessary to note, that last time some papers have been issued, where QM with GUR is considered at the very beginning. As a consequence of this approach an stable remnants due to the process of Black Hole evaporation appears.

On the other hand from results obtained above, qualitatively we can answer to the question about information loss on the black holes, which are formed when a star collapses. Indeed, near to the horizon of events an approximately pure state has an entropy practically equal to zero: $S^{in} = -Sp[\rho \ln(\rho)]$, which corresponds to the value $\beta \rightarrow 0$. When it is approaching to a singularity $\beta > 0$ (in other words to the Planck scale) and it has yet a non equal to zero entropy: $S_\beta = -Sp[\rho(\beta) \ln(\rho(\beta))]$. Therefore in a black hole entropy increases as well as information is lost.

6 Conclusion

There is a question. Is it rightful to use the commonly defined measurement procedure in Quantum Gravity? So far in many papers on Quantum gravity (see for instance [16]) any other procedure has not been used or proposed. But as it was shown above in the case when Quantum Gravity effects are important there are not pure states. And the other hand as it was noted in [17] all known approaches to justify Quantum Gravity one way or another lead to the notion of fundamental length. Besides that GUR (1), which as well lead to that notion are well described within the inflation model [18]. Therefore, apparently is not possible to understand physics at the Planck's scale without these notions. Besides that, it is necessary to consider one more aspect of this problem. As it was noted in [19], when a new physical theory is created, it implies the introduction of a new parameter and the deformation of precedent theory by this parameter. All these deformation parameters are in their essence fundamental constants: G , c and \hbar (more exactly in [19] instead of c , $1/c$ is used). From the results presented above it follows, that the question formulated in [19] can be specified: is it this theory, the theory with fundamental length, which contains by definition all these three parameters: $L_p = \sqrt{\frac{G\hbar}{c^3}}$?

This paper is the extended and revised version of [20].

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